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Coherent states and their time dependence in fractional dimensions

A Thilagam and M A Lohe

Department of Physics, The University of Adelaide 5005, Australia

E-mail: Max.Lohe@adelaide.edu.au

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Abstract
We construct representations of the Lie algebra $\mathfrak{su}(1, 1)$ using representations of the momentum and position operators satisfying the R-deformed Heisenberg relations, in which the fractional dimension d and angular momentum ℓ appear as parameters. The Bargmann index κ, which characterizes representations of the positive discrete series of $\mathfrak{su}(1, 1)$, can take any positive value. We construct coherent states in fractional dimensions, in particular we extend the two well-known analytic representations of coherent states for $\mathfrak{su}(1, 1)$, Perelomov and Barut–Girardello states, from dimension one to any dimension d. We generalize this construction to time-dependent coherent states by means of the $\mathfrak{su}(1, 1)$ symmetries of the quantum time-dependent harmonic oscillator in fractional dimensions. We investigate the uncertainty relations of the momentum and position operators with respect to these coherent states, and their dependence on the dimension.

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1. Introduction

Fractional dimensions have been used in a variety of studies, ranging from calculations of critical exponents of Ising models in arbitrary dimensions [1] to the modelling of refractive index [2], donor states [3] and superconductivity [4] in solid state systems. Fractional dimensional models are efficient in dealing with complicated problems that are otherwise analytically intractable, requiring extensive computational efforts [5–7].

Recently, following work of Stillinger [8], we formulated [9] an algebraic approach to the quantum description of physical models in fractional dimensions in which the dimension d and angular momentum ℓ appear as parameters within representations of the position and momentum operators. Quadratic combinations of these operators satisfy an $\mathfrak{su}(1, 1)$ algebra which can be used to solve the harmonic oscillator for any d. Here we extend this algebraic approach to the construction of coherent states. Such states define the limit between classical and nonclassical behaviour of quantum states and possess probability distributions which follow classical trajectories. Coherent states are therefore suitable for the investigation of quasiclassical features of a quantum mechanical process. Numerous studies related to coherent states of one-dimensional oscillator systems have appeared in the literature [10, 11] including several reviews [12–14].

A general method of constructing standard coherent states is by means of a unitary displacement operator that acts on a reference state which may be chosen to be the vacuum [15, 16]. Another involves the construction of an eigenstate of the annihilation operator [10, 16] while a third identifies states which yield the minimum uncertainty [17]. Coherent states of parabosons, which are relevant to our approach, can be constructed using the direct method of Sharma *et al* [18], which defines coherent states as eigenstates of the annihilation operator. We discuss this method within the context of fractional dimensions. The second method, which has been well investigated for the one-dimensional harmonic oscillator [19], exploits properties of the $\mathfrak{su}(1, 1)$ algebra, and we also extend this to fractional dimensions.

The method of dynamical invariants introduced by Lewis and Riesenfeld [20] provides a means of finding exact solutions to the time-dependent Schrödinger equation, which in [9] we used to solve the time-dependent harmonic oscillator for any d and any (integer) ℓ. We use these wavefunctions and properties of the $\mathfrak{su}(1, 1)$ algebra to construct corresponding time-dependent coherent states which, to our knowledge, have not been previously derived. These states display a nontrivial time dependence and provide an example of 'temporal stability' discussed by Gazeau and Klauder [21], in which a system initially in a coherent state evolves as a coherent state at all later times. The Lie algebra $\mathfrak{su}(1, 1)$ is of interest in quantum optics as it characterizes various quantum optical systems. It is well known [19, 22] that the bosonic realization of $\mathfrak{su}(1, 1)$ gives an accurate description of degenerate and non-degenerate parametric amplifiers. Here, we investigate squeezing properties of coherent states of the harmonic oscillator in fractional dimensions. The study of squeezed states (i.e. reduced fluctuations for one observable to a value less than its ground state value) is interesting because, other than giving us new understanding of quantum phenomena, there is potential application for reducing noise in optical and communication systems [23].

In section 2 we review properties of the harmonic oscillator in fractional dimensions and in section 3 we summarize properties of the relevant $\mathfrak{su}(1, 1)$ algebra. In section 4, we generalize analytic representations of coherent states for $\mathfrak{su}(1, 1)$ in one dimension, namely Perelomov and Barut–Girardello states, to fractional dimensions, and we also reformulate the direct method [18] of constructing coherent states for parabosons. In section 5, we investigate uncertainty relations and squeezing properties of coherent states in fractional dimensions, particularly with respect to the dimensional dependence. In section 6, we discuss solutions to the time-dependent harmonic oscillator in dimension d and construct corresponding coherent states. We also construct coherent states associated with the analytic representations of the associated $\mathfrak{su}(1, 1)$ algebra and discuss their properties.

2. The harmonic oscillator in fractional dimensions

The algebraic formulation of quantum mechanics for one degree of freedom, in any dimension $d > 0$, begins with the following relations proposed by Wigner [24] in connection with quantization of the harmonic oscillator:

$$[Q^2, P] = 2\mathrm{i}Q, \qquad [P^2, Q] = -2\mathrm{i}P, \tag{1}$$

where P, Q are the momentum and position operators. As discussed in [9], these relations may be written more conveniently by introducing the reflection operator R to read

$$[Q, P] = \mathrm{i}(1 + \nu R), \qquad \{Q, R\} = 0 = \{P, R\}, \qquad R^2 = 1 \tag{2}$$

where ν is a real parameter. We also have the Hermiticity properties

$$Q^* = Q, \qquad P^* = P, \qquad R^* = R.$$

If we define annihilation and creation operators $a, a^\dagger$ in the usual way according to

$$a = \tfrac{1}{\sqrt{2}}(Q + \mathrm{i}P), \qquad a^\dagger = \tfrac{1}{\sqrt{2}}(Q - \mathrm{i}P),$$

then $a, a^\dagger$ satisfy the trilinear relations of a paraboson algebra with one degree of freedom:

$$[\{a, a^\dagger\}, a] = -2a, \qquad [\{a, a^\dagger\}, a^\dagger] = 2a^\dagger,$$

or, in more convenient form,

$$[a, a^\dagger] = 1 + \nu R, \qquad \{a, R\} = 0 = \{a^\dagger, R\}, \qquad R^2 = 1. \tag{3}$$

The paraboson order is $1 \pm \nu$ (depending on the parity of the vacuum on which the states are built) but, in contradistinction to the case of general paraboson algebras (as discussed by Greenberg and Messiah [25]), is not restricted to integer values only. The commutation relations (2) and (3) appeared in [26] (equations (2.24) and (2.25)) but without the dimensional interpretation we develop here.

The connection with quantum mechanics in fractional dimensions arises from the following coordinate representation of P, Q, R, as described in [9]. Let P, Q, R act in a Hilbert space $\mathfrak{H}$ of complex functions $\psi(x)$ defined on $\mathbb{R}$ with an inner product defined by

$$(\psi, \phi) = \int_{-\infty}^{\infty} |x|^{d-1}\overline{\psi(x)}\phi(x)\,\mathrm{d}x, \tag{4}$$

with

$$\begin{aligned} R\psi(x) &= \psi(-x) \\ Q\psi(x) &= x\psi(x) \\ P\psi(x) &= \left[-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{i}\nu}{2}x^{-1}R - \frac{\mathrm{i}(d-1)}{2}x^{-1}\right]\psi(x), \end{aligned} \tag{5}$$

then the relations (2) are satisfied. The operators P, Q, R are each Hermitian, provided that $\psi(x)$ has suitable behaviour at infinity and at the origin, and so may be extended to self-adjoint operators on $\mathfrak{H}$. The parameter ν is chosen such that $P^2 = -\Delta_{\mathrm{radial}}$ within $\mathfrak{H}$, where the radial Laplacian is defined by

$$\Delta_{\mathrm{radial}} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{(d-1)}{x}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\ell(\ell + d - 2)}{x^2}, \tag{6}$$

where the parameter ℓ may be identified with angular momentum and takes the values $\ell = 0, 1, 2\ldots.$ The dimension d can take any positive value. We assume that the Hamiltonian under consideration has the form $H = \frac{1}{2}[P^2 + V(Q)]$ where the potential V is an even function of Q and hence commutes with R, and in this case the eigenfunctions of H are either even or odd. Since the parity of the eigenfunction depends on whether ℓ is even or odd, R has the eigenvalue $(-1)^\ell$. We also identify

$$\nu = (-1)^\ell(d - 1 + 2\ell) \tag{7}$$

but due to the invariance of Δ_{radial} under $\ell \longrightarrow -\ell - d + 2$ another choice is

$$\nu = (-1)^{\ell-1}(d - 3 + 2\ell), \tag{8}$$

which corresponds to the previous case with $\ell \to \ell - 1$. In each case we have $P^2\psi = -\Delta_{\rm radial}\psi$ as required, for all eigenfunctions ψ of H. The Hilbert space $\mathfrak{H}$ under consideration decomposes into the direct sum of subspaces $\mathfrak{H}_\ell$, each carrying the label ℓ, i.e. $\mathfrak{H} = \Sigma_\ell \oplus \mathfrak{H}_\ell$.

Let us consider now specifically the harmonic oscillator, for which the Hamiltonian is

$$H = \tfrac{1}{2}(P^2 + Q^2) = \tfrac{1}{2}(a^\dagger a + a a^\dagger) \tag{9}$$

and for which, as is well known (see for example [9, 26, 27]), the eigenfunctions are

$$|n\rangle_{\rm e} = N_n^{-\frac{1}{2}}(a^\dagger)^n|0\rangle_{\rm e}, \tag{10}$$

where $|0\rangle_{\rm e}$ is the even vacuum state with the properties $a|0\rangle_{\rm e} = 0$ and $R|0\rangle_{\rm e} = |0\rangle_{\rm e}$, and where the normalization is given by

$$N_n = \prod_{k=1}^{n}\left[k + \frac{\nu}{2}(1 - (-1)^k\right].$$

Specifically we have, for even and odd subscripts,

$$N_{2m} = 2^{2m} m!\left(\frac{1+\nu}{2}\right)_m \qquad N_{2m+1} = 2^{2m+1} m!\left(\frac{1+\nu}{2}\right)_{m+1},$$

where m is an integer, and where we have used the Pochhammer product notation

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(n)}, \tag{11}$$

where n can be a positive or negative integer. The eigenfunctions (10) are even or odd, according to whether the paraboson number n is even or odd. It follows that

$$a^\dagger|2m\rangle_{\rm e} = \sqrt{2m+1+\nu}\,|2m+1\rangle_{\rm e}, \qquad a^\dagger|2m+1\rangle_{\rm e} = \sqrt{2m+2}\,|2m+2\rangle_{\rm e}, \tag{12}$$

with similar matrix elements between states built on the odd vacuum $|0\rangle_{\rm o}$. The eigenvalues of H corresponding to the states built on the even vacuum are

$$E_n = n + \tfrac{1}{2}(\nu + 1), \qquad n = 0, 1, 2\ldots \tag{13}$$

while for those built on the odd vacuum ν is replaced by $-\nu$ as follows from the symmetry $R \to -R$.

The specific application to quantum mechanics in fractional dimensions is obtained by substituting the explicit representation (5) for P, Q, R, and hence also for the paraboson operators a, $a^\dagger$. In the coordinate representation the eigenfunctions $\psi_{m,\ell}(x)$ of the Hamiltonian are solutions of the equation

$$\tfrac{1}{2}(-\Delta_{\rm radial} + x^2)\psi = E\psi$$

and take the form derived by Stillinger [8] (normalized with respect to the inner product (4)):

$$\psi_{m,\ell}(x) = \langle x|m\rangle_\ell = \sqrt{\frac{m!}{\Gamma\left(m+\ell+\frac{d}{2}\right)}}(-1)^m x^\ell\,{\rm e}^{-\frac{1}{2}x^2} L_m^{(\ell-1+\frac{d}{2})}(x^2), \tag{14}$$

where L denotes generalized Laguerre polynomials, and where m is a nonnegative integer. Properties of Laguerre polynomials and their connection with $\mathfrak{su}(1,1)$ have been discussed and summarized by Biedenharn and Louck [30] (see Topic 6 in [30], pp 284, 304).

The functions $\psi_{m,\ell}$ are even or odd according to whether ℓ is even or odd, and d can take any positive value. The energy levels are given by

$$E_{m,\ell} = 2m + \ell + \frac{d}{2}, \qquad m = 0, 1, 2 \dots . \tag{15}$$

The simple harmonic oscillator is regained for $d = 1$ by choosing either $\ell = 0$ (even states) or $\ell = 1$ (odd states). In this case the generalized Laguerre polynomials reduce to the usual Hermite polynomials, see [28] (p 779).

By solving the equation $a|0\rangle = 0$ we find that the vacuum state for each fixed ℓ is $\langle x|0\rangle_\ell = \psi_{0,\ell}(x)$ which is therefore the unique (up to normalization) vacuum state within each subspace $\mathfrak{H}_\ell$ of $\mathfrak{H}$. The paraboson order for states built on the vacuum in $\mathfrak{H}_\ell$ is $d + 2\ell$.

Neither a nor $a^\dagger$ act within $\mathfrak{H}_\ell$ for fixed ℓ since these operators do not preserve parity. We find, choosing $\nu = (-1)^\ell(d - 1 + 2\ell)$, that

$$a^\dagger\psi_{m,\ell} = \sqrt{2m + 2\ell + d}\,\psi_{m,\ell+1} \qquad a^\dagger\psi_{m,\ell+1} = \sqrt{2m + 2}\,\psi_{m+1,\ell} \tag{16}$$

in accordance with (12). In each case the energy level (15) is raised by one unit, as indicated also by the energy levels (13). Evidently, in the first case the creation operator increments the angular momentum ℓ by addition of paraboson quanta.

The operators $(a^\dagger)^2$ and a^2 act within each subspace $\mathfrak{H}_\ell$ and behave as a raising (lowering) operator for the energy levels given in (15). The properties of these operators, which generate $\mathfrak{su}(1, 1)$, are considered in the next section. Hence, even powers of $a^\dagger$ or a act within $\mathfrak{H}_\ell$, and odd powers act as mappings $\mathfrak{H}_\ell \longrightarrow \mathfrak{H}_{\ell\pm1}$.

3. $\mathfrak{su}(1, 1)$ symmetry

Fundamental to our formulation is the existence of symmetries, in particular the invariance of the relations (2) under $SL_2(\mathbb{R})$ transformations of P, Q. The corresponding Lie algebra $\mathfrak{sl}_2(\mathbb{R}) \sim \mathfrak{su}(1, 1)$ is generated by quadratic combinations of P, Q, specifically

$$K_0 = \tfrac{1}{4}(Q^2 + P^2), \qquad K_1 = \tfrac{1}{4}(Q^2 - P^2), \qquad K_2 = -\tfrac{1}{4}(QP + PQ), \tag{17}$$

for which

$$[K_1, K_2] = -\mathrm{i}K_0, \qquad [K_0, K_1] = \mathrm{i}K_2, \qquad [K_0, K_2] = -\mathrm{i}K_1.$$

The relations (1) state that the pair (P, Q) forms a spinor operator with respect to this algebra, as discussed in [29].

In terms of the paraboson operators the algebra $\mathfrak{su}(1, 1)$ is generated by $\{K_0, K_\pm = K_1 \pm \mathrm{i}K_2\}$ defined by

$$K_0 = \tfrac{1}{4}(a^\dagger a + aa^\dagger), \qquad K_+ = \tfrac{1}{2}{a^\dagger}^2 \qquad K_- = \tfrac{1}{2}a^2, \tag{18}$$

which satisfy

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = -2K_0. \tag{19}$$

Irreducible unitary representations of this algebra are infinite dimensional and for the case at hand belong to the positive discrete series, as discussed for example by Biedenharn and Louck [30] (p 276), also Perelomov [15] (p 70), and Kastrup [31]. States in the representation space are either even or odd and are labelled by eigenvalues of the mutually commuting generators R, K_0, C where C is the Casimir invariant:

$$C = K_0^2 - \tfrac{1}{2}(K_-K_+ + K_+K_-) = K_0(K_0 - 1) - K_+K_-.$$

This operator C does not commute with either a or $a^\dagger$ however, following [32], we may define $\widetilde{C} = C + \frac{1}{8}[a, a^\dagger]$ which is independent of R and commutes with these paraboson operators. The operator $\widetilde{C}$ can be realized in the Lie superalgebra $osp(1|2)$ generated by $K_\pm, K_0, V_+ = \sqrt{\frac{1}{8}}a^\dagger$ and $V_- = -\sqrt{\frac{1}{8}}a^\dagger$, see [32].

As is well known [33] the orthonormal basis $\{|\kappa, m\rangle_\pm\}$, where $m = 0, 1, 2, \ldots,$ satisfies

$$
\begin{aligned}
R|\kappa, m\rangle_\pm &= \pm|\kappa, m\rangle_\pm \\
C|\kappa, m\rangle_\pm &= \kappa(\kappa - 1)|\kappa, m\rangle_\pm \\
K_0|\kappa, m\rangle_\pm &= (\kappa + m)|\kappa, m\rangle_\pm \\
K_+|\kappa, m\rangle_\pm &= \sqrt{(m+1)(m+2\kappa)}|\kappa, m+1\rangle_\pm \\
K_-|\kappa, m\rangle_\pm &= \sqrt{m(m+2\kappa-1)}|\kappa, m-1\rangle_\pm
\end{aligned}
\tag{20}
$$

where the Bargmann index κ is any positive number (for unitary representations of the covering group of $SU(1,1)$). The relation of κ to ν is determined by evaluating C using the realization (18):

$$C = \tfrac{1}{16}(\nu R - 3)(\nu R + 1),$$

leading to

$$C|\kappa, m\rangle_- = \tfrac{1}{16}(\nu + 3)(\nu - 1)|\kappa, m\rangle_- \qquad C|\kappa, m\rangle_+ = \tfrac{1}{16}(\nu - 3)(\nu + 1)|\kappa, m\rangle_+.$$

We may identify the Bargmann indices κ_{e} and κ_{o} for the even and odd subspaces respectively as

$$\kappa_{\mathrm{e}} = \tfrac{1}{4}(1 + \nu), \qquad \kappa_{\mathrm{o}} = \tfrac{1}{4}(3 + \nu), \tag{21}$$

and using (20) and (21) we can demonstrate the equivalence

$$|\kappa_{\mathrm{e}}, m\rangle_+ \equiv |2m\rangle_{\mathrm{e}}, \qquad |\kappa_{\mathrm{o}}, m\rangle_- \equiv |2m+1\rangle_{\mathrm{e}}, \tag{22}$$

where the Fock states on the right, defined by (10), implicitly carry the label ν. There is a similar equivalence for Fock states built on the odd vacuum $|0\rangle_{\mathrm{o}}$, with $\nu \to -\nu$.

At $\nu = 0$ we regain the well-known values $\kappa_{\mathrm{e}} = \frac{1}{4}$ and $\kappa_{\mathrm{o}} = \frac{3}{4}$ for the usual harmonic oscillator (see [15] and also [19]). For general ν, however, there are no restrictions on the possible values of κ, and we obtain all representations of the positive discrete series of $\mathfrak{su}(1,1)$.

In order to establish properties of coherent states in fractional dimensions it is convenient to express the $\mathfrak{su}(1,1)$ generators directly in terms of the realization given by (5). We have then

$$K_0 = \frac{1}{4}(-\Delta_{\mathrm{radial}} + x^2) \qquad K_1 = \frac{1}{4}(\Delta_{\mathrm{radial}} + x^2) \qquad K_2 = \frac{\mathrm{i}}{2}\left(x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{d}{2}\right), \tag{23}$$

where Δ_{radial} is defined in (6). The Casimir invariant simplifies to

$$C = \tfrac{1}{16}(d + 2\ell)(d + 2\ell - 4)I$$

and hence the Bargmann index is given by

$$\kappa = \tfrac{1}{4}(d + 2\ell), \tag{24}$$

and so takes any positive value as d, ℓ are varied. Since we require $\kappa > 0$ we discard in general the other possible value $\kappa = \frac{1}{4}(-d - 2l + 4)$ which results from the invariance under $\ell \longrightarrow -\ell - d + 2$. This corresponds to the fact that under this symmetry the eigenfunctions (14) are transformed into functions that in general do not belong to the required Hilbert space $\mathfrak{H}$. The special cases $\kappa = \frac{1}{4}, \frac{3}{4}$ are regained with $d = 1, \ell = 0, 1$, and similarly for the second choice for κ.

The matrix elements of $\{K_0, K_\pm\}$ are given by (20) in which the states $|\kappa, m\rangle_\pm$ are associated with the eigenfunctions $\psi_{m,\ell}(x)$ defined in (14), which are even or odd according to the value of ℓ. The generators $\{K_0, K_\pm\}$ act within the subspace $\mathfrak{H}_\ell$ of $\mathfrak{H}$, preserving both parity and angular momentum, and therefore provide a convenient means for constructing coherent states.

4. Coherent states in fractional dimensions

Coherent states and their properties have been widely studied, see for example the volume of collected papers [11] (edited by Klauder and Skagerstam) also [10], and the recent review article by Vourdas [14], which contains many further references. Coherent states for paraboson operators have also been extensively investigated, see for example the papers by Sudarshan and collaborators [18, 26, 34]. A brief study of coherent states in fractional dimensions was undertaken in [35] for $\nu = d - 1$ with the angular momentum values restricted to $\ell = 0$ and $\ell = 1$.

For the simple harmonic oscillator in one dimension, coherent states may be defined using properties of the displacement operator, see for example the discussion in Klauder and Sudarshan [26], chapter 7, and although displacement transformations can also be implemented for the R-deformed relations (2), they do not have the simple properties which follow from the canonical commutation relations, essentially because the relations (1) are not invariant under simple translations of Q or P. We describe in turn three other methods by which coherent states may be defined in fractional dimensions, firstly as eigenstates of the annihilation operator, and then also using analytic representations of $\mathfrak{su}(1, 1)$.

4.1. Coherent states as eigenstates of the annihilation operator

Coherent states may be defined as eigenstates of the annihilation operator:

$$a|\alpha\rangle_\ell = \alpha|\alpha\rangle_\ell \tag{25}$$

where $\alpha \in \mathbb{C}$, and where we have indicated the dependence on ℓ. The states $|\alpha\rangle_\ell$ may be expanded as a linear superposition of the paraboson states $|n\rangle$ shown in (10), as has been done for paraboson states in [18].

In the application to quantum mechanics in fractional dimensions we replace the paraboson states by the explicit wavefunctions $\psi_{m,\ell}(x)$ defined in (14) and hence expand the states $\langle x|\alpha\rangle_\ell$ (in the coordinate representation) as follows:

$$\langle x|\alpha\rangle_\ell = \sum_{m=0}^{\infty} (b_m \psi_{m,\ell} + c_m \psi_{m,\ell+1}) \tag{26}$$

for some coefficients $\{b_m, c_m\}$, where we have allowed for an expansion over both even and odd states. The action of the annihilation operator on these states, with $\nu = (-1)^\ell (d - 1 + 2\ell)$, is given by

$$a\psi_{m,\ell} = \sqrt{2m}\,\psi_{m-1,\ell+1}, \qquad a\psi_{m,\ell+1} = \sqrt{2m + 2\ell + d}\,\psi_{m,\ell}$$

as follows from (16). Hence, we obtain

$$b_m = \frac{C_\ell \alpha^{2m}}{\sqrt{m! 2^{2m+2\kappa-1}\Gamma(2\kappa + m)}}, \qquad c_m = \frac{C_\ell \alpha^{2m+1}}{\sqrt{m! 2^{2m+2\kappa}\Gamma(2\kappa + m + 1)}}, \tag{27}$$

where C_ℓ is a normalization constant, and $2\kappa = \ell + \frac{d}{2}$. The normalization ${}_\ell\langle\alpha|\alpha\rangle_\ell = 1$ leads to

$$\sum_{m=0}^{\infty}(|b_m|^2 + |c_m|^2) = 1,$$

as follows from orthonormality of the eigenfunctions $\psi_{m,\ell}$ with respect to m and also with respect to eigenfunctions of opposite parity. Define the function

$$F_\mu(z) = I_{\mu-1}(z) + I_\mu(z), \tag{28}$$

where I_μ denotes a modified Bessel function of order μ, of the first kind, then we obtain C_ℓ in the form

$$C_\ell = \frac{|\alpha|^{2\kappa-1}}{\sqrt{F_{2\kappa}(|\alpha|^2)}}. \tag{29}$$

We may now sum the series (26) by using one of the generating functions for generalized Laguerre polynomials (see [28] p 784) to obtain

$$\langle x|\alpha\rangle_\ell = \left(\frac{\alpha}{|\alpha|}\right)^{1-2\kappa} \mathrm{e}^{-\frac{1}{2}(x^2+\alpha^2)} x^{1-\frac{d}{2}} \frac{F_{2\kappa}(\sqrt{2}x\alpha)}{\sqrt{F_{2\kappa}(|\alpha|^2)}}. \tag{30}$$

As an example, for the special case of the simple harmonic oscillator ($d = 1$, $\ell = 0$) we obtain the well-known expression

$$\langle x|\alpha\rangle_{\ell=0} = \pi^{-\frac{1}{4}} \mathrm{e}^{-\frac{1}{2}|\alpha|^2} \mathrm{e}^{-\frac{1}{2}(x^2+\alpha^2)+\sqrt{2}x\alpha}.$$

A more direct method of finding the eigenfunctions $\langle x|\alpha\rangle_\ell$ is to solve (25) as a differential equation by means of the representation (5) for P, Q. If we denote $u(x) = \langle x|\alpha\rangle_\ell$ then

$$u' + \left(x - \frac{\nu}{2x}R + \frac{d-1}{2x}\right)u = \sqrt{2}\alpha u$$

and since $u = u_\mathrm{e} + u_\mathrm{o}$ (writing u as a sum of even and odd parts) we obtain

$$u_\mathrm{e}' + \left(x - \frac{\ell}{x}\right)u_\mathrm{e} = \sqrt{2}\alpha u_\mathrm{o} \qquad u_\mathrm{o}' + \left(x + \frac{d-1+\ell}{x}\right)u_\mathrm{o} = \sqrt{2}\alpha u_\mathrm{e}.$$

This leads to second-order differential equations for u_e and u_o, each of which have a solution that is regular at the origin and is expressible in terms of modified Bessel functions. The sum of these two solutions gives u and hence the coherent state $\langle x|\alpha\rangle_\ell$ with the precise x-dependence as shown in (30).

4.2. Coherent states using analytic representations of $\mathfrak{su}(1,1)$

There are two well-known analytic representations of coherent states for $\mathfrak{su}(1,1)$ which have been studied in quantum optics for single model fields at $\nu = 0$. The first are the Perelomov generalized coherent states [15] which are obtained using the displacement operator formalism and to which we refer as a Perelomov $\mathfrak{su}(1,1)$ coherent state. The second $\mathfrak{su}(1,1)$ coherent states are based on the overcomplete basis of the Barut–Girardello coherent states [33] which are the analogue of harmonic oscillator coherent states, namely eigenstates of the annihilation operator.

The standard Perelomov coherent states $|\zeta,\kappa\rangle_{\mathrm{P}}$ are obtained (see [15], chapter 5, also [22]) by allowing the unitary operator $\mathrm{e}^{\xi K_+-\bar{\xi}K_-}$ to act on the vacuum:

$$\begin{aligned}|\zeta,\kappa\rangle_{\mathrm{P}} &= \mathrm{e}^{\xi K_+-\bar{\xi}K_-}|\kappa,0\rangle \\ &= (1-|\zeta|^2)^{\kappa}\sum_{m=0}^{\infty}\sqrt{\frac{\Gamma(m+2\kappa)}{m!\Gamma(2\kappa)}}\zeta^m|\kappa,m\rangle\end{aligned} \tag{31}$$

where $\zeta = \frac{\xi}{|\xi|}\tanh|\xi|$ satisfies $|\zeta| < 1$. Equation (31) describes the well-known squeezed states [36] which are equivalent to two-photon coherent states [37]. Properties of the squeezing operator $S(\xi) = \mathrm{e}^{\xi K_+-\bar{\xi}K_-}$ have been discussed by Stoler [36].

We now explicitly evaluate these coherent states $|\zeta,\kappa\rangle_{\mathrm{P}}$ in the coordinate representation by substituting the value $\kappa = \frac{1}{4}(d+2\ell)$ derived in (24), which is appropriate to our formulation in fractional dimensions, and by replacing the eigenstates $|\kappa,m\rangle$ by the normalized wavefunctions $\psi_{m,\ell}(x)$ given in (14). The sum over m can be performed explicitly with the help of the generating function for generalized Laguerre polynomials (see [28], p 784) to obtain

$$\langle x|\zeta,\kappa\rangle_{\mathrm{P}} = \frac{x^{\ell}(1-|\zeta|^2)^{\kappa}}{\sqrt{\Gamma(2\kappa)}(1+\zeta)^{2\kappa}}\exp\left[\frac{x^2(\zeta-1)}{2(\zeta+1)}\right], \tag{32}$$

where $2\kappa = \ell + \frac{d}{2}$. In this form we see directly that the functions $\langle x|\zeta,\kappa\rangle_{\mathrm{P}}$ are defined everywhere on the complex plane, except on the line $\zeta = -1$, but are normalizable as functions of x only in the unit disc $|\zeta| < 1$. These states are known [14, 22] to resolve the identity operator for $\kappa > \frac{1}{2}$, by integration over the unit disc. The overlap of two Perelomov coherent states takes the form

$${}_{\mathrm{P}}\langle\zeta_1,\kappa|\zeta_2,\kappa\rangle_{\mathrm{P}} = \frac{(1-|\zeta_1|^2)^{\kappa}(1-|\zeta_2|^2)^{\kappa}}{(1-\overline{\zeta_1}\zeta_2)^{2\kappa}}, \tag{33}$$

as may be verified directly from the expression (32). In contrast to the states (30), the functional form of the states (32) does not change as d varies.

Barut–Girardello coherent states $|z,\kappa\rangle_{\mathrm{BG}}$ are defined [33] as eigenstates of the lowering operator K_-:

$$K_-|z,\kappa\rangle_{\mathrm{BG}} = z|z,\kappa\rangle_{\mathrm{BG}} \tag{34}$$

and have the expression

$$|z,\kappa\rangle_{\mathrm{BG}} = \frac{z^{\kappa-\frac{1}{2}}}{\sqrt{I_{2\kappa-1}(2|z|)}}\sum_{m=0}^{\infty}\frac{z^m}{\sqrt{m!\Gamma(m+2\kappa)}}|\kappa,m\rangle \tag{35}$$

where z is an arbitrary complex number, and $I_{2\kappa-1}$ denotes a modified Bessel function of the first kind. (In the notation of Barut and Girardello [33] we have $E_0 = -\Phi = \kappa$ and their $\sqrt{2}z$ is replaced by our z.) Various properties of Barut–Girardello states including the completeness relations and their overlap have been discussed in [22, 14] (and other references cited therein).

As in the case of Perelomov states, we can explicitly evaluate the states $\langle x|z,\kappa\rangle_{\mathrm{BG}}$ by substituting $\kappa = \frac{1}{4}(d+2\ell)$ and replacing the eigenstates $|\kappa,m\rangle$ by the normalized wavefunctions $\psi_{m,\ell}(x)$ given in (14). As before, the sum over m can be performed explicitly with the help of the generating function for generalized Laguerre polynomials ([28], p 784) to obtain

$$\langle x|z,\kappa\rangle_{\mathrm{BG}} = x^{1-\frac{d}{2}}\,\mathrm{e}^{-z-\frac{1}{2}x^2}\frac{I_{\ell+\frac{d}{2}-1}(2x\sqrt{z})}{\sqrt{I_{\ell+\frac{d}{2}-1}(2|z|)}}, \tag{36}$$

where we have displayed explicitly the dependence on d. Alternatively, we may solve (34) directly by using the expressions (23) for K_1, K_2 to obtain $u(x) = \langle x|z, \kappa\rangle_{\mathrm{BG}}$ as the normalized solution (regular at the origin) of the differential equation

$$\Delta_{\mathrm{radial}} u + 2xu' + (d + x^2)u = 4zu.$$

Both the states $\langle x|z, \kappa\rangle_{\mathrm{BG}}$ and the Perelomov states $\langle x|\zeta, \kappa\rangle_{\mathrm{P}}$ defined in (32) are eigenfunctions of R, i.e. they have parity $(-1)^\ell$ and so are even or odd as functions of x according to whether ℓ is even or odd. The overlap between Barut–Girardello states and Perelomov states is known to be [14, 22]

$$_{\mathrm{P}}\langle \zeta, \kappa|z, \kappa\rangle_{\mathrm{BG}} = \frac{z^{\kappa-1/2}(1 - |\zeta|^2)^\kappa \exp(\overline{\zeta} z)}{\sqrt{I_{2\kappa-1}(2|z|)\Gamma(2\kappa)}} \tag{37}$$

which may be verified explicitly from the expressions (32) and (36).

The relation between Barut–Girardello states and Perelomov states is further discussed in [14, 22]. The coherent states (30) which are defined as the eigenfunctions $|\alpha\rangle_\ell$ of the annihilation operator, where $\alpha = \sqrt{2z}$, are also related to the Barut–Girardello states as a linear combination, as follows from $a^2|\alpha\rangle_\ell = \alpha^2|\alpha\rangle_\ell$.

5. Uncertainty relations in fractional dimensions

We now investigate the way in which the dimension affects uncertainty relations of various quantum mechanical operators. Any pair of Hermitian operators Ω, Λ satisfies the Schwartz inequality (see, for example, [38], chapter 9)

$$(\Delta\Omega)^2(\Delta\Lambda)^2 \geqslant \tfrac{1}{4}\langle\{\widehat{\Omega}, \widehat{\Lambda}\}\rangle^2 + \tfrac{1}{4}\langle -\mathrm{i}[\Omega, \Lambda]\rangle^2 \geqslant \tfrac{1}{4}\langle -\mathrm{i}[\Omega, \Lambda]\rangle^2, \tag{38}$$

where $\widehat{\Omega} = \Omega - \langle\Omega\rangle$, in which $\langle\Omega\rangle$ denotes the average or expectation value of Ω with respect to a given reference state, and the uncertainty $\Delta\Omega$ is defined by

$$(\Delta\Omega)^2 = \langle(\widehat{\Omega})^2\rangle = \langle\Omega^2\rangle - \langle\Omega\rangle^2.$$

If we assume that the reference state carries definite angular momentum ℓ, then $\langle\nu R\rangle = d - 1 + 2\ell$ as follows from (7). Hence,

$$\Delta Q \Delta P \geqslant \tfrac{1}{2}|\langle -\mathrm{i}[Q, P]\rangle| = \tfrac{1}{2}\langle 1 + \nu R\rangle = \tfrac{1}{2}(d + 2\ell) \geqslant \tfrac{1}{2}d. \tag{39}$$

As we show below, equality is achieved for the ground state of the harmonic oscillator. For $\ell = 0$ we deduce that the uncertainty can be reduced to an arbitrarily small value, by choosing sufficiently small $d > 0$. The expression (39) reduces to the expected form at $d = 1$ and $\ell = 0$ but differs from the uncertainty relations derived for the specific case of $\ell = 0$ in [35] where separate expressions for the odd and even states were obtained.

Equation (39) can be generalized to the Robertson–Schrödinger uncertainty relation [39] by using (38) to obtain

$$\langle P^2\rangle\langle Q^2\rangle - \tfrac{1}{4}\langle QP + PQ\rangle^2 \geqslant \tfrac{1}{4}\langle -\mathrm{i}[Q, P]\rangle^2 = \tfrac{1}{4}(d + 2\ell)^2. \tag{40}$$

The left-hand side is invariant under canonical transformations of the commutation relations (2), that is under linear $SL(2)$ transformations of the spinor pair (P, Q) and implies the Heisenberg uncertainty relation (39). A similar relation follows from (38) for the uncertainties ΔP, ΔQ involving $\widehat{P}$, $\widehat{Q}$.

5.1. Average taken with respect to harmonic oscillator states

By taking the average with respect to the normalized states $\psi_{m,\ell}(x)$ defined in (14) we obtain

$$\langle P\rangle = 0 = \langle Q\rangle, \qquad \langle P^2\rangle = \langle Q^2\rangle = \tfrac{1}{2}(4m+d+2\ell),$$

and therefore

$$\Delta Q\Delta P = \sqrt{\langle P^2\rangle\langle Q^2\rangle} = \tfrac{1}{2}(4m+d+2\ell),$$

and so we attain the lower bound shown in (39) at $m = 0$.

We can extend this uncertainty relation to higher-order moments of P and Q by calculating $\langle P^r\rangle$ and $\langle Q^r\rangle$ for any integer r. We have, firstly,

$$\langle P^r\rangle = 0 = \langle Q^r\rangle \qquad (\text{odd } r),$$

since in this case the integrand in the inner product (4) is odd.

In order to calculate $\langle P^{2n}\rangle$ and $\langle Q^{2n}\rangle$ for integers n we use properties of the $\mathfrak{su}(1,1)$ algebra generated by quadratic combinations of P, Q as described in section 3. We have

$$P^2 = 2K_0 - K_+ - K_-, \qquad Q^2 = 2K_0 + K_+ + K_-, \tag{41}$$

and so we define

$$\Gamma = 2K_0 + \eta(K_+ + K_-),$$

where $\eta^2 = 1$, and now determine the matrix element $\langle\Gamma^n\rangle$ for any integer n.

The matrix elements of the generators of $\mathfrak{su}(1,1)$ are given by (20), where $\kappa = \frac{1}{4}(d+2\ell)$, and hence

$$\Gamma|\kappa,m\rangle = 2(\kappa+m)|\kappa,m\rangle + \eta\sqrt{(m+1)(m+2\kappa)}|\kappa,m+1\rangle + \eta\sqrt{m(m+2\kappa-1)}|\kappa,m-1\rangle,$$

which implies, by induction on n,

$$\Gamma^n|\kappa,m\rangle = \sum_{i=0}^{2n} A_i^n(m)\sqrt{(m+1)_{i-n}(m+2\kappa)_{i-n}}|\kappa,m-n+i\rangle$$

for any integer $n \geqslant 0$, where we have used the Pochhammer notation defined in (11). The coefficients $A_i^n(m)$, which are zero unless $0 \leqslant i \leqslant 2n$, satisfy the recurrence relation

$$\begin{aligned} A_i^n(m) = {} & \eta(m-n+i+1)(2\kappa+m-n+i)A_i^{n-1}(m) \\ & + 2(\kappa+m-n+i)A_{i-1}^{n-1}(m) + \eta A_{i-2}^{n-1}(m), \end{aligned} \tag{42}$$

starting with $A_0^0(m) = 1$. In particular, we wish to determine the coefficients $A_n^n(m)$, for then we obtain $\langle\Gamma^n\rangle = A_n^n(m)$.

We find that $A_n^n(m)$ is a polynomial of degree n in the quantum number m, with positive coefficients, hence $A_n^n(m) \geqslant A_n^n(0)$. For $m = 0$ the solution to the recurrence relations (42) is

$$A_i^n(0) = \begin{cases} \eta^{i-n}\begin{pmatrix} n \\ i-n \end{pmatrix}(2\kappa+i-n)_{2n-i} & n \leqslant i \leqslant 2n \\ 0 & \text{otherwise} \end{cases}$$

as may be verified directly by substitution in (42).

Hence, for any positive integer n we obtain

$$\langle P^{2n}\rangle = \langle Q^{2n}\rangle = A_n^n(m) \geqslant A_n^n(0) = (2\kappa)_n = \left(\tfrac{1}{2}(d+2\ell)\right)_n$$

with equality when we choose the ground state $m = 0$. For $2\kappa = \frac{1}{2}$, i.e. $d = 1$, $\ell = 0$ this result agrees with that found by Santhanam ([40], equation (14)). We may now calculate uncertainties $\Delta(P^r)\Delta(Q^r)$ for any integers r, for example if r is odd we obtain

$$\Delta(P^r)\Delta(Q^r) = \sqrt{\langle P^{2r}\rangle\langle Q^{2r}\rangle} \geqslant (2\kappa)_r.$$

Similarly,

$$\Delta(P^{2n})\Delta(Q^{2n}) = A_{2n}^{2n}(m) - \left(A_n^n(m)\right)^2 \geqslant (2\kappa)_n[(2\kappa+n)_n - (2\kappa)_n],$$

for any integer n. For the case $n = 2$ we have

$$\Delta(P^4)\Delta(Q^4) \geqslant 4\kappa(2\kappa+1)(4\kappa+3),$$

with equality for the ground state $m = 0$, which is the minimum uncertainty involving the fourth moments. Again, for $\ell = 0$ these uncertainties are proportional to the dimension d and so are arbitrarily small for sufficiently small d.

5.2. *Average taken with respect to the coherent state $|\alpha\rangle_\ell$*

We now investigate the uncertainty relations in which the average is taken with respect to the states $|\alpha\rangle_\ell$, defined in (25) as the eigenstates of the annihilation operator. These states minimize the uncertainty, i.e. the general inequality shown in (39) holds with equality due to the relation

$$(\widehat{Q} + \mathrm{i}\widehat{P})|\alpha\rangle_\ell = 0,$$

but we will verify directly that $\Delta Q\Delta P = \frac{1}{2}|\langle -\mathrm{i}[Q,P]\rangle| = \frac{1}{2}\langle 1 + \nu R\rangle$ for these coherent states. Uncertainty relations for paraboson coherent states have been studied earlier [18, 26], but without a dimensional interpretation.

First, we evaluate the average value $\langle R\rangle$ using (26)

$$\langle R\rangle = {}_\ell\langle\alpha|R|\alpha\rangle_\ell = (-1)^\ell\sum_{m=0}^{\infty}(|b_m|^2 - |c_m|^2)$$
$$= (-1)^\ell\frac{I_{2\kappa-1}(|\alpha|^2) - I_{2\kappa}(|\alpha|^2)}{I_{2\kappa-1}(|\alpha|^2) + I_{2\kappa}(|\alpha|^2)}. \tag{43}$$

From (3) we find

$$\langle [a, a^\dagger]\rangle = 1 + (-1)^\ell(d - 1 + 2\ell)\langle R\rangle \tag{44}$$

and, since $\langle a^\dagger a\rangle = |\alpha|^2$, we deduce

$$\langle\{a, a^\dagger\}\rangle = 1 + (-1)^\ell(d - 1 + 2\ell)\langle R\rangle + 2|\alpha|^2. \tag{45}$$

The uncertainty $(\Delta Q)^2 = \langle Q^2\rangle - \langle Q\rangle^2$ may be calculated using

$$\langle Q\rangle = \tfrac{1}{\sqrt{2}}\langle a + a^\dagger\rangle = \tfrac{1}{\sqrt{2}}(\alpha + \overline{\alpha}) \qquad \langle Q^2\rangle = \tfrac{1}{2}\left(\alpha^2 + \overline{\alpha}^2\right) + \tfrac{1}{2}\langle\{a, a^\dagger\}\rangle,$$

and hence

$$(\Delta Q)^2 = \tfrac{1}{2} + \tfrac{1}{2}(-1)^\ell(d - 1 + 2\ell)\langle R\rangle.$$

We determine ΔP similarly and find $\Delta P = \Delta Q$, and so

$$\Delta Q\Delta P = \tfrac{1}{2}[1 + (-1)^\ell(d - 1 + 2\ell)\langle R\rangle] = \tfrac{1}{2}|\langle -\mathrm{i}[Q,P]\rangle|,$$

as expected. Numerical calculations show that squeezing (i.e. $\Delta Q\Delta P < \frac{1}{2}$) takes place over a range of values of d and $|\alpha|$ with $\ell = 0, 1$. For small $|\alpha|$ and fixed κ we have the expansion

$$\Delta Q\Delta P = 2\kappa + \left(\frac{1}{4\kappa} - 1\right)|\alpha|^2 + O(|\alpha|^4),$$

showing that the uncertainty is 2κ for small $|\alpha|$, consistent with (39).

5.3. Average taken with respect to Perelomov and Barut–Girardello states

Since the Perelomov and Barut–Girardello states (32) and (36) have definite parity we have in each case the average values $\langle P\rangle = 0 = \langle Q\rangle$. For the Perelomov states we may calculate averages directly using the expression (32) to obtain

$$\langle P^2\rangle_{\mathrm{P}} = \frac{1}{2}(d+2\ell)\frac{(1-\zeta)(1-\bar{\zeta})}{1-\zeta\bar{\zeta}}, \qquad \langle Q^2\rangle_{\mathrm{P}} = \frac{1}{2}(d+2\ell)\frac{(1+\zeta)(1+\bar{\zeta})}{1-\zeta\bar{\zeta}},$$

and so we have the uncertainty

$$\Delta Q\Delta P = \frac{1}{2}(d+2\ell)\frac{\sqrt{(1-\zeta^2)(1-\overline{\zeta^2})}}{1-\zeta\bar{\zeta}}, \tag{46}$$

which reduces to (39) for real or pure imaginary values of ζ with $|\zeta| < 1$.

In the case of the Barut–Girardello states (36) we first calculate (following [33]) the average value $\langle K_0\rangle_{\mathrm{BG}}$, where K_0 is the $\mathfrak{su}(1,1)$ generator with the matrix elements as shown in (20). From (35), using orthonormality of the basis $\{|\kappa, m\rangle\}$, we obtain

$$\begin{aligned}\langle K_0\rangle_{\mathrm{BG}} = {}_{\mathrm{BG}}\langle z,\kappa|K_0|z,\kappa\rangle_{\mathrm{BG}} &= \frac{|z|^{2\kappa-1}}{I_{2\kappa-1}(2|z|)}\sum_{m=0}^{\infty}\frac{|z|^{2m}(m+\kappa)}{m!\Gamma(m+2\kappa)} \\ &= \kappa + \frac{|z|I_{2\kappa}(2|z|)}{I_{2\kappa-1}(2|z|)}.\end{aligned} \tag{47}$$

From (41) we have

$$\langle P^2\rangle_{\mathrm{BG}} = 2\langle K_0\rangle_{\mathrm{BG}} - z - \bar{z}, \qquad \langle Q^2\rangle_{\mathrm{BG}} = 2\langle K_0\rangle_{\mathrm{BG}} + z + \bar{z}, \tag{48}$$

from which we calculate the uncertainty $\Delta Q\Delta P = \sqrt{\langle P^2\rangle_{\mathrm{BG}}\langle Q^2\rangle_{\mathrm{BG}}}$ with respect to Barut–Girardello states which, like (46), is evidently a function only of $\kappa = \frac{1}{4}(d+2\ell)$ and the complex parameter, in this case $z, \bar{z}$. In contrast to (46), however, the uncertainty does not reduce to (39) for real or pure imaginary z. For small $|z|$ and fixed κ we have the expansion

$$\Delta Q\Delta P = 2\kappa + \frac{|z|^2\sin^2\theta}{\kappa} + O(|z|^4),$$

where $z = |z|\,\mathrm{e}^{\mathrm{i}\theta}$, showing that the uncertainty is arbitrarily small for small κ and sufficiently small $|z|$.

6. Time-dependent harmonic oscillator

We now investigate time-dependent coherent states in any dimension d. First we describe exact solutions of the time-dependent harmonic oscillator, which in fractional dimensions is defined by the Hamiltonian

$$H(t) = \tfrac{1}{2}(P^2 + \omega(t)^2Q^2), \tag{49}$$

where P, Q satisfy (1) and (2), and where the frequency $\omega(t)$ is a given function of time. In order to solve the time-dependent Schrödinger equation

$$\left[\mathrm{i}\frac{\partial}{\partial t} - H(t)\right]\psi(x,t) = 0, \tag{50}$$

we use the method of Lewis and Riesenfeld [20], originally developed in one dimension. We define a dynamical invariant $I(t)$ which satisfies

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial t} + \mathrm{i}[H, I] = 0. \tag{51}$$

I has eigenvalues which are constant in time and its time-dependent eigenfunctions, multiplied by a phase factor, solve (50). If H can be expressed as a linear combination of elements of a Lie algebra, then I may also be constructed as a linear combination of these same elements, with time-dependent coefficients. In our case, the relevant algebra is $\mathfrak{su}(1,1)$ generated by quadratic combinations of P, Q, as shown in (17).

As described in our previous paper [9], following Lewis and Riesenfeld [20], the invariant I has the expression

$$\begin{aligned} I &= \tfrac{1}{2}\rho^2 P^2 + \tfrac{1}{2}(\rho^{-2} + \dot{\rho}^2)Q^2 - \tfrac{1}{2}\rho\dot{\rho}(PQ + QP) \\ &= \tfrac{1}{2}(a^\dagger a + a a^\dagger), \end{aligned} \tag{52}$$

where the time-dependent operators $a(t)$, $a^\dagger(t)$ are given by

$$a^\dagger = \frac{1}{\sqrt{2}}((\rho^{-1} + \mathrm{i}\dot{\rho})Q - \mathrm{i}\rho P), \qquad a = \frac{1}{\sqrt{2}}((\rho^{-1} - \mathrm{i}\dot{\rho})Q + \mathrm{i}\rho P), \tag{53}$$

and satisfy the equal time paraboson relations analogous to (3)

$$[a, a^\dagger] = 1 + \nu R, \qquad \{a, R\} = \{a^\dagger, R\} = 0, \qquad [a, I] = a, \qquad [a^\dagger, I] = -a^\dagger. \tag{54}$$

The function $\rho(t)$ is any solution of the Ermakov equation [41]

$$\ddot{\rho} + \omega(t)^2\rho = \frac{1}{\rho^3}, \tag{55}$$

all solutions of which can be constructed (following [42, 43, 20]) from solutions $f(t)$ of the linear equation of motion for the classical time-dependent harmonic oscillator:

$$\ddot{f} + \omega(t)^2 f = 0. \tag{56}$$

Specifically, the general solution of (55) is given by

$$\rho^2 = f_1^2 + W^{-2} f_2^2,$$

where f_1, f_2 are linearly independent solutions of (56), and where the Wronskian $W = f_1\dot{f}_2 - \dot{f}_1 f_2$ is a nonzero constant.

It follows from the form of this general solution that $\rho(t)^2$ is strictly positive for all t (see appendix 2 in [44] and the discussion in appendix B of [45]), and hence $\rho(t) > 0$ for all t. Arbitrary powers of ρ, as appear in the time-dependent wavefunctions (see (62)), are therefore well defined, as is also the function $\Omega(t)$ defined in (58).

Given any solution f of (56), a second linearly independent solution is $f(t)\int\frac{\mathrm{d}t}{f(t)^2}$, which leads to the following convenient formula for solutions of the Ermakov equation (55):

$$\rho(t)^2 = f(t)^2\left[1 + \left(\int\frac{\mathrm{d}t}{f(t)^2}\right)^2\right], \tag{57}$$

as may be verified directly. The general solution depends on two constants, a multiplicative constant associated with f and an integration constant.

The eigenvalues of I can be determined exactly as for the case of the time-independent Hamiltonian because I has the same form as H, as shown in (9). The eigenstates $|n,t\rangle_I$ of I are given by (10), where the creation operators are now time dependent, and I has eigenvalues as given by (13). We will construct these eigenstates, however, more directly as wavefunctions $\phi_{m,\ell}(x,t)$ in the Hilbert space $\mathfrak{H}$.

In the Heisenberg picture, the time dependence of the operators $a^\dagger(t)$, $a(t)$ is given [9] by

$$a^\dagger(t) = \mathrm{e}^{\mathrm{i}\Omega(t)}\,a_0^\dagger, \qquad a(t) = \mathrm{e}^{-\mathrm{i}\Omega(t)}\,a_0$$

where $a_0 = a(0)$, $a_0^\dagger = a^\dagger(0)$ and

$$\Omega(t) = \int_0^t \frac{\mathrm{d}t'}{\rho(t')^2}, \tag{58}$$

taking $t = 0$ as the initial time. These relations imply that

$$I(t) = I(0) = \tfrac{1}{2}\left(a_0^\dagger a_0 + a_0 a_0^\dagger\right),$$

which explicitly demonstrates that I is constant in time.

Also, as in the time-independent case, the operators $K_0(t)$, $K_+(t)$ and $K_-(t)$ generate $\mathfrak{su}(1,1)$, where

$$\begin{aligned}
K_0(t) &= \tfrac{1}{4}\{a(t), a^\dagger(t)\} = \tfrac{1}{4}\left\{a_0^\dagger, a_0\right\} = K_0(0),\\
K_+(t) &= \tfrac{1}{2}a^\dagger(t)^2 = \tfrac{1}{2}\,\mathrm{e}^{2\mathrm{i}\Omega(t)}\left(a_0^\dagger\right)^2 = \mathrm{e}^{2\mathrm{i}\Omega(t)}\,K_+(0)\\
K_-(t) &= \tfrac{1}{2}a(t)^2 = \tfrac{1}{2}\,\mathrm{e}^{-2\mathrm{i}\Omega(t)}(a_0)^2 = \mathrm{e}^{-2\mathrm{i}\Omega(t)}\,K_-(0),
\end{aligned}$$

and satisfy (19) at any given time. Representations are given by (20) and are labelled by the index κ as shown in (21) which, in the application to fractional dimensions, takes the value $\kappa = \frac{1}{4}(d + 2\ell)$. Evidently I is related to K_0 by $I(t) = 2K_0(t)$ and has the eigenvalues $2\kappa + 2m$ for $m = 0, 1, 2, \ldots$.

A useful observation by several authors (see for example [46]), which relies on properties of $\mathfrak{su}(1,1)$ and therefore extends to the fractional dimensional case, is that $I(t)$ and the time-independent Hamiltonian are related by a simple unitary transformation, specifically

$$T I(t) T^\dagger = \tfrac{1}{2}(P^2 + Q^2), \tag{59}$$

where

$$T = \exp\left(\mathrm{i}\frac{\log\rho}{2}(QP + PQ)\right)\exp\left(-\mathrm{i}\frac{\dot\rho}{2\rho}Q^2\right). \tag{60}$$

Hence, the time-dependent eigenstates $|n, t\rangle_I$ of I are given by

$$|n, t\rangle_I = T^\dagger |n\rangle_{\mathrm{e,o}}$$

where $|n\rangle_{\mathrm{e,o}}$ denotes the even or odd time-independent states as defined in (10). The application of this construction to quantum mechanics in fractional dimensions is obtained in the coordinate representation as for the time-independent case, by substituting the explicit representation (5) for P, Q, R. This leads to the construction of the explicit wavefunctions $\phi_{m,\ell}(x, t)$, as given in [9] and as shown in (64).

The time-dependent Schrödinger equation reads

$$H\psi(x,t) = \tfrac{1}{2}[-\Delta_{\mathrm{radial}} + \omega(t)^2 x^2]\psi(x,t) = \mathrm{i}\frac{\partial}{\partial t}\psi(x,t), \tag{61}$$

where Δ_{radial} is defined by (6). The solutions $\psi_{m,\ell}(x,t)$ are obtained (following [20, 9]) by multiplying the corresponding eigenfunctions $\phi_{m,\ell}(x,t)$ of I by a phase factor $\mathrm{e}^{\mathrm{i}\alpha_m(t)}$ where

$$\alpha_m(t) = -\left(2m + \ell + \frac{d}{2}\right)\Omega(t),$$

where $\Omega(t)$ is defined by (58). The normalized solutions $\psi_{m,\ell}(x,t)$ of (61) are then given by

$$\begin{aligned}
\psi_{m,\ell}(x,t) = {}& \sqrt{\frac{m!}{\Gamma\left(m+\ell+\frac{d}{2}\right)}}(-1)^m L_m^{\left(\ell-1+\frac{d}{2}\right)}\left(\frac{x^2}{\rho^2}\right)x^\ell \rho^{-\frac{1}{2}(d+2\ell)}\\
&\times \exp\left[-\frac{x^2}{2\rho}(\rho^{-1} - \mathrm{i}\dot\rho)\right]\exp\left[-\mathrm{i}\left(2m + \ell + \frac{d}{2}\right)\Omega\right].
\end{aligned} \tag{62}$$

These wavefunctions, which are well defined for any $d > 0$ and any $\ell = 0, 1, 2, \ldots$, are orthonormal for fixed ℓ, at any time t, with respect to the inner product (4):

$$(\psi_{m,\ell}, \psi_{m',\ell}) = \delta_{m,m'},$$

and are also orthogonal for wavefunctions of opposite parity.

For constant $\rho^2 = \omega^{-1}$ these wavefunctions are separable as functions of x and t, and reduce to those for the time-independent case shown in (14), multiplied by a phase factor that determines their time evolution. Even for constant ω, however, solutions with nontrivial time dependence can be obtained by choosing nontrivial solutions of the Ermakov equation (55). For example, a family of solutions, depending on a parameter λ, is

$$\rho(t)^2 = \frac{\cos^2 \omega t + \lambda^4 \sin^2 \omega t}{\omega\lambda^2}, \tag{63}$$

corresponding to the choice

$$f(t) = \frac{1}{\lambda\sqrt{\omega}} \cos \omega t$$

in (56) and (57). In this case $\Omega(t) = \arctan\left(\lambda^2 \tan \omega t\right)$ and the wavefunctions $\psi_{m,\ell}(x,t)$ have a nontrivial time dependence for any $\lambda \neq 1$. The wavefunction evolves from an initial configuration which is a rescaled pure state and so develops a nontrivial time dependence. Such solutions lead to time-dependent coherent states, even for constant ω, as we now investigate.

6.1. Time-dependent coherent states

We first construct time-dependent coherent states $|\alpha, t\rangle_\ell$ as eigenfunctions of the time-dependent annihilation operator $a(t)$,

$$a(t)|\alpha, t\rangle_\ell = \alpha|\alpha, t\rangle_\ell,$$

following the calculation in section 4.1. We therefore expand $\langle x|\alpha, t\rangle_\ell$ in terms of eigenfunctions of I which, instead of the wavefunctions $\psi_{m,\ell}(x,t)$ shown in (62), we take to be the following orthonormal wavefunctions $\phi_{m,\ell}(x,t)$, omitting for convenience the phase factor $\mathrm{e}^{\mathrm{i}\alpha_m(t)}$:

$$\phi_{m,\ell}(x,t) = \sqrt{\frac{m!}{\Gamma(m+2\kappa)}}(-1)^m L_m^{(2\kappa-1)}\left(\frac{x^2}{\rho^2}\right) x^\ell \rho^{-2\kappa} \exp\left[-\frac{x^2}{2\rho}(\rho^{-1} - \mathrm{i}\dot{\rho})\right], \tag{64}$$

where $2\kappa = \ell + \frac{d}{2}$. We have

$$\langle x|\alpha, t\rangle_\ell = \sum_{m=0}^{\infty} b_m \phi_{m,\ell}(x,t) + c_m \phi_{m,\ell+1}(x,t), \tag{65}$$

and using

$$a\phi_{m,\ell} = \sqrt{2m}\phi_{m-1,\ell+1} \qquad a\phi_{m,\ell+1} = \sqrt{2m+4\kappa}\phi_{m,\ell}$$

we obtain the same coefficients b_m, c_m as shown in (27), together with the normalization as shown in (29).

We may sum the series (65) as before to obtain

$$\langle x|\alpha, t\rangle_\ell = \rho^{-1} x^{1-\frac{d}{2}} \left(\frac{\alpha}{|\alpha|}\right)^{-2\kappa+1} \mathrm{e}^{-\frac{\alpha^2}{2}} \exp\left[-\frac{x^2}{2\rho}(\rho^{-1} - \mathrm{i}\dot{\rho})\right] \frac{F_{2\kappa}\left(\frac{\sqrt{2}x\alpha}{\rho}\right)}{\sqrt{F_{2\kappa}(|\alpha|^2)}}, \tag{66}$$

where F is defined in (28). As an example, for the special case of the simple harmonic oscillator $\left(d = 1, \kappa = \frac{1}{4}\right)$ we obtain

$$\langle x|\alpha, t\rangle_{\ell=0} = \frac{1}{\sqrt{\pi^{\frac{1}{2}}\rho}} \mathrm{e}^{-\frac{1}{2}(\alpha^2+|\alpha|^2)} \mathrm{e}^{\frac{\sqrt{2}x\alpha}{\rho}} \exp\left[-\frac{x^2}{2\rho}(\rho^{-1} - \mathrm{i}\dot{\rho})\right].$$

The construction of coherent states using analytic representations of $\mathfrak{su}(1,1)$ proceeds as for the time-independent case, due to the fact that the matrix elements of the generators of $\mathfrak{su}(1,1)$ remain time independent, as shown in (20). We derive the time-dependent Perelomov coherent states $\langle x|\zeta, t, \kappa\rangle_{\mathrm{P}}$ from the expression (31) by replacing the eigenstates $|\kappa, m\rangle$ by the associated wavefunctions $\phi_{m,\ell}(x, t)$ given in (64). After using the same generating function for generalized Laguerre polynomials as before we obtain

$$\langle x|\zeta, t, \kappa\rangle_{\mathrm{P}} = \frac{x^{\ell}(1-|\zeta|^2)^{\kappa}}{\rho^{2\kappa}\sqrt{\Gamma(2\kappa)}(1+\zeta)^{2\kappa}} \exp\left(\mathrm{i}\frac{\dot{\rho}x^2}{2\rho}\right) \exp\left[\frac{x^2(\zeta-1)}{2\rho^2(\zeta+1)}\right]. \tag{67}$$

The overlap of two time-dependent Perelomov coherent states is time independent and has the same expression as before, see (33). This is due to the fact that the time dependence of the coherent states (67) appears only through a rescaling of the x-variable by the function ρ, together with a phase factor and a multiplicative factor which cancel in the integration.

As in the case of Perelomov states, we can explicitly evaluate the Barut–Girardello states $|z, t, \kappa\rangle_{\mathrm{BG}}$ in the coordinate representation by substituting $\kappa = \frac{1}{4}(d + 2\ell)$ and replacing the eigenstates $|\kappa, m\rangle$ by the normalized wavefunctions $\phi_{m,\ell}(x)$ given in (64). As before, the sum over m can be performed explicitly with the help of the generating function for generalized Laguerre polynomials to obtain

$$\langle x|z, t, \kappa\rangle_{\mathrm{BG}} = \rho^{-1}x^{1-\frac{d}{2}}\,\mathrm{e}^{-z} \exp\left[-\frac{x^2}{2\rho}(\rho^{-1} - \mathrm{i}\dot{\rho})\right] \frac{I_{2\kappa-1}\left(\frac{2x\sqrt{z}}{\rho}\right)}{\sqrt{I_{2\kappa-1}(2|z|)}}. \tag{68}$$

The overlap of these states is also time independent, in the same manner as occurs for the Perelomov states, and similarly for the overlap between Barut–Girardello states and Perelomov states as shown in (37).

6.2. *Uncertainty relations with respect to time-dependent coherent states*

Next, we investigate the uncertainty relations associated with the time-dependent oscillator with respect to various states. By taking the average with respect to the normalized states defined in (62) or (64) we obtain $\langle P\rangle = 0 = \langle Q\rangle$, and also the time-dependent values

$$\langle P^2\rangle = \tfrac{1}{2}(4m + d + 2\ell)(\rho^{-2} + \dot{\rho}^2) \qquad \langle Q^2\rangle = \tfrac{1}{2}(4m + d + 2\ell)\rho^2.$$

Therefore,

$$\Delta Q \Delta P = \sqrt{\langle P^2\rangle\langle Q^2\rangle} = \tfrac{1}{2}(4m + d + 2\ell)\sqrt{1 + \rho^2\dot{\rho}^2}, \tag{69}$$

and so the uncertainty always increases with respect to these time-dependent states.

It is also of interest to evaluate the expectation value of the Hamiltonian H with respect to these same wavefunctions; we find

$$\langle H\rangle = \tfrac{1}{4}(4m + d + 2\ell)(\rho^{-2} + \dot{\rho}^2 + \omega^2\rho^2), \tag{70}$$

where the right-hand side is time independent if and only if ω is time independent. For the specific solutions ρ given by (63), corresponding to constant ω, we find

$$\langle H\rangle = \tfrac{1}{4}(4m + d + 2\ell)(\lambda^{-2} + \lambda^2)\omega$$

which, as expected, takes its minimum value at $\lambda = 1$. The expression (70) shows that the energy levels of H remain equally spaced as time varies, as observed also by Lewis [20] for $d = 1$.

Next, we evaluate the uncertainty with respect to the time-dependent coherent states $|\alpha, t\rangle_\ell$ derived in (66), which are eigenfunctions of the annihilation operator. $\langle R\rangle$ is time independent and so is given by (43), and similarly for the average values $\langle[a, a^\dagger]\rangle$ and $\langle\{a, a^\dagger\}\rangle$, which are given by (44) and (45), respectively. $(\Delta Q)^2$ and $(\Delta P)^2$ may be calculated using

$$Q = \tfrac{1}{\sqrt{2}}\rho(a + a^\dagger), \qquad P = \tfrac{1}{\sqrt{2}}(-\mathrm{i}\rho^{-1} + \dot\rho)a + \tfrac{1}{\sqrt{2}}(\mathrm{i}\rho^{-1} + \dot\rho)a^\dagger,$$

leading to

$$(\Delta Q)^2 = \tfrac{1}{2}\rho^2\,(1 + \nu\langle R\rangle) \qquad (\Delta P)^2 = \tfrac{1}{2}(\rho^{-2} + \dot\rho^2)(1 + \nu\langle R\rangle),$$

where $\nu = (-1)^\ell(d - 1 + 2\ell)$. Hence,

$$\Delta Q\Delta P = \tfrac{1}{2}\sqrt{1 + \rho^2\dot\rho^2}(1 + \nu\langle R\rangle)$$

where $\langle R\rangle$ is given by (43).

The uncertainty with respect to the Perelomov and Barut–Girardello states (67) and (68) may be calculated as before. We have the average values $\langle P\rangle = 0 = \langle Q\rangle$, and for the Perelomov states we find

$$\langle P^2\rangle_{\mathrm{P}} = \frac{1}{2}(d + 2\ell)\frac{[(1-\zeta)\rho^{-1} - \mathrm{i}(1+\zeta)\dot\rho][(1-\bar\zeta)\rho^{-1} + \mathrm{i}(1+\bar\zeta)\dot\rho]}{1 - \zeta\bar\zeta}$$

$$\langle Q^2\rangle_{\mathrm{P}} = \frac{1}{2}(d + 2\ell)\rho^2\frac{(1+\zeta)(1+\bar\zeta)}{1 - \zeta\bar\zeta},$$

and so we obtain

$$\Delta Q\Delta P = \frac{1}{2}(d + 2\ell)\frac{\sqrt{[(1-\zeta^2) - \mathrm{i}(1+\zeta)^2\rho\dot\rho][(1-\bar\zeta^2) + \mathrm{i}(1+\bar\zeta)^2\rho\dot\rho]}}{1 - \zeta\bar\zeta}.$$

For $\zeta = 0$ this reduces to the previous uncertainty (69) at $m = 0$, corresponding to the fact that in this case the Perelomov states (67) necessarily reduce to the wavefunctions $\phi_{m,\ell}(x, t)$ given in (64). The uncertainty for the Barut–Girardello states (68) is time independent and therefore has the same value as derived from (47) and (48). This occurs because the matrix elements of the $\mathfrak{su}(1, 1)$ generators are time independent and hence, according to (41), so also are the expectation values of P^2 and Q^2.

7. Conclusion

We have extended analytic representations of coherent states for $\mathfrak{su}(1, 1)$, the Perelomov and Barut–Girardello states, from one dimension to any fractional dimension d. We have obtained closed form expressions for these coherent states which has enabled us to investigate their properties as functions of d. By means of dynamical invariants we have constructed time-varying harmonic oscillator wavefunctions and corresponding time-dependent coherent states in fractional dimensions, again in closed form, including coherent states associated with analytic representations of $\mathfrak{su}(1, 1)$, and investigated their properties as functions of time and dimension. Finally, we have examined uncertainty relations and the specific role of dimensionality with regard to squeezing properties of coherent states.

References

[1] Wilson K G and Fisher M E 1972 *Phys. Rev. Lett.* **28** 240–3
[2] Tanguy C, Lefebvre P, Mathieu H and Elliott R J 1997 *J. Appl. Phys.* **82** 798–802
[3] Mikhailov I D, Betancur F J, Escorcia R A and Sierra-Ortega J 2003 *Phys. Rev.* B **67** 115317
[4] Bak Z 2003 *Phys. Rev.* B **68** 064511
[5] He X F 1991 *Phys. Rev.* B **43** 2063–69
[6] Thilagam A 1997 *Phys. Rev.* B **56** 4665–70
[7] Syed M, Yang G L, Furdyna J K, Dobrowolska M and Kossut J 2001 *Superlattices Microstruct.* **29** 247–57
[8] Stillinger F H 1977 *J. Math. Phys.* **18** 1224–34
[9] Lohe M A and Thilagam A 2004 *J. Phys. A: Math. Gen.* **37** 6181–99
[10] Klauder J R and Sudarshan E C G 1968 *Fundamentals of Quantum Optics* (New York: Benjamin)
[11] Klauder J R and Skagerstam B S 1985 *Coherent States, Applications in Physics and Mathematical Physics* (Singapore: World Scientific)
[12] Zhang W M, Feng D H and Gilmore R 1990 *Rev. Mod. Phys.* **62** 867–927
[13] Dodonov V V 2002 *Quantum Semiclass. Opt.* **4** R1–R33
[14] Vourdas A 2006 *J. Phys. A: Math. Gen.* **39** R65–R141
[15] Perelomov A 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
[16] Glauber R J 1963 *Phys. Rev.* **131** 2766–88
[17] Nieto M M and Simmons L M Jr 1978 *Phys. Rev. Lett.* **41** 207–10
[18] Sharma J K, Mehta C L and Sudarshan E C G 1978 *J. Math. Phys.* **19** 2089–93
[19] Bishop R F and Vourdas A 1998 *J. Phys. A: Math. Gen.* **31** 9341–52
[20] Lewis H R 1968 *J. Math. Phys.* **9** 1976–86
Lewis H R and Riesenfeld W B 1969 *J. Math. Phys.* **10** 1458–73
[21] Gazeau J P and Klauder J R 1999 *J. Phys. A: Math. Gen.* **32** 123–32
[22] Brif C, Vourdas A and Mann A 1996 *J. Phys. A: Math. Gen.* **29** 5873–85
[23] Scully M O and Zubary M S 1996 *Quantum Optics* (Cambridge: Cambridge University Press)
[24] Wigner E P 1950 *Phys. Rev.* **77** 711–2
[25] Greenberg O W and Messiah A M L 1965 *Phys. Rev.* **138** B1155–67
[26] Mukunda N, Sudarshan E C G, Sharma J K and Mehta C L 1980 *J. Math. Phys.* **21** 2386–94
[27] Jordan T F, Mukunda N and Pepper S V 1963 *J. Math. Phys.* **4** 1089–95
[28] Abramowitz M and Stegun I (ed) 1965 *Handbook of Mathematical Functions* (New York: Dover)
[29] Lohe M A and Thilagam A 2005 *J. Phys. A: Math. Gen.* **38** 461–83
[30] Biedenharn L C and Louck J D 1981 *Racah–Wigner Algebra in Quantum Theory (Encyclopedia of Mathematics and its Applications* vol 9) (Reading, MA: Addison-Wesley)
[31] Kastrup H A 2003 *Fortsch. Phys.-Prog. Phys.* **51** 975–1134
[32] Brzeziński T, Equsquiza I L and Macfarlane A J 1993 *Phys. Lett.* B **311** 202–6
[33] Barut A O and Girardello L 1971 *Commun. Math. Phys.* **21** 41–55
[34] Sharma J K, Mehta C L, Mukunda N and Sudarshan E C G 1981 *J. Math. Phys.* **22** 78–90
[35] Matos-Abiague A 2001 *J. Phys. A: Math. Gen.* **34** 11059–68
[36] Stoler D 1970 *Phys. Rev.* D **1** 3217–9
[37] Yuen H P 1976 *Phys. Rev.* A **13** 2226–43
[38] Shankar R 1980 *Principles of Quantum Mechanics* (New York: Plenum)
[39] Roberston H P 1929 *Phys. Rev.* **34** 163–4
[40] Santhanam T S 2000 *J. Phys. A: Math. Gen.* **33** L83–5
[41] Ermakov V P 1880 *Univ. Izv. Kiev* **20** 1–25
[42] Milne E W 1930 *Phys. Rev.* **35** 863–7
[43] Pinney E 1950 *Proc. Am. Math. Soc.* **1** 681
[44] Nunez J A, Bensch F and Korsch H J 1991 *J. Phys. A: Math. Gen.* **24** 2069–81
[45] Fernando Barbero G J, Vergel D G and Villaseñor E J S 2006 *Phys. Rev.* D **74** 024003
[46] Maamache M 1998 *J. Math. Phys.* **39** 161–9
Fernández Guasti M and Moya-Cessa H 2003 *J. Phys. A: Math. Gen.* **36** 2069–76
[47] Hartley J G and Ray J R 1982 *Phys. Rev.* D **25** 382–6
[48] Pedrosa I A 1987 *Phys. Rev.* D **36** 1279–80